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TEACHER VERSION
MODELING THE STEEPING OF SOUTHERN SWEET ICED TEA

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Abstract: We offer raw data collected from a webcam and a thermometer for evaluating the strength of steeping tea. We ask students to build a mathematical model using the data to predict how long the tea should steep before essentially reaching saturation.

Keywords: Sweet Tea, Newton's Law of Cooling, modeling, first order

Tags: ordinary, differential equations, parameter estimation, data

STATEMENT

There are many preferences among Southerners regarding iced tea. Some prefer unsweetened iced tea, but most prefer sweet tea. Among those that prefer sweet tea, the preferred sweetness varies. Some like to sweeten their iced tea with artificial sweeteners. Among those that use sugar to sweeten their tea, some like lightly sweetened iced tea, others like moderately sweetened, and still others like their tea extremely sweet. That is, the preference of sugar concentration is probably the biggest difference between different sweet teas.

Another difference between iced teas is the type of tea used in brewing. There are instant teas, iced tea mixes, flavored teas, decaffeinated teas, cold brew tea bags, and classic fresh brew tea bags. Yet another preference Southerners have among their ice teas is the brand of tea. Some of the most common brands are Lipton,® Luzianne,® Red Diamond,® and Tetley.®

Figure 1. Lipton® Family Size Tea Bags

Figure 2. Manufacturer's brewing directions

The following directions are used by the author when brewing sweet tea. The number of tea bags used and brewing time are the primary differences between these and the manufacturer's directions.

Directions
Bring $9\frac{1}{2}$ cups of water to a boil. Pour boiling water onto $1\frac{1}{3}$ cups of granulated sugar, and stir until the sugar has dissolved. Bob two Lipton® family size tea bags (see Figures 1 and 2 for package images) in the water several (10-12) times. Let the tea steep for about an hour (or so). Remove the tea bags, and stir in 8 cups of cold water. Refrigerate for several hours (if possible) before serving.

Note: If the directions above are followed, then the total amount of water used is $17\frac{1}{2}$ cups (which is about 1.1 gallons), and thus the tea pitcher used must be able to sustain this capacity. It is also worth noting that the sugar concentration used is about 1.22 (or essentially $1\frac{1}{4}$) cups/gallon.

Question 1: *How long should the tea brew so that its concentration essentially reaches steady state?*

Question 2: *How does this steady state concentration compare to the concentration attained using twice the number of tea bags brewed for the manufacturer's suggested 3-5 minutes?*

In order to answer these questions, we must determine a way to quantify the concentration level of the tea. Since the color of the brewed tea gets darker the longer the tea brews, we assume that the tea concentration stabilizes when the *darkness* of the brewed tea stabilizes. To that end, we monitor the *darkness* of the tea by periodically taking photos of the brewing tea throughout the steeping process.

Data Collection

We begin our data collection by using a spotlight positioned on one side of the tea pitcher and a webcam on the opposite side. The Linux command `uvccapture` is used to periodically (i.e. every 5 seconds) take snapshots of the steeping tea using a Logitech HD Pro Webcam C910. Figure 3 shows the initial frame captured by the webcam, and demonstrates that the pitcher initially contains the sugar and no water.

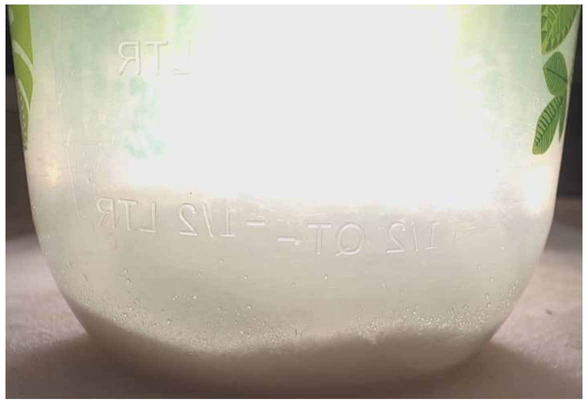

Figure 3. Initial Frame Captured by Webcam

A custom Perl script was written to processes a portion of an image and return a quadruplet of the form (R,G,B,L) representing the red/green/blue/luminance of this region. The output RGBL quadruplet is the average red/green/blue/luminance values of each pixel within the circular region of the image with provided center and radius. The relative luminance, L, is determined by the standard [1, p. 4] formula

$$L = 0.2126 \cdot R + 0.7152 \cdot G + 0.0722 \cdot B$$

where R, G, and B, are the red, green, and blue components, respectively. These collected R, G, and B values, as well as the computed L values, are illustrated in Figure 4 with colors red, green, blue, and black, respectively.

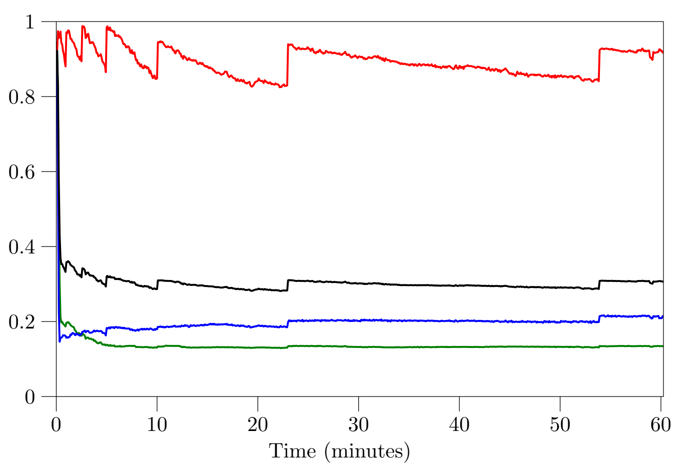

Figure 4. Steeping RGBL Values

Comparing the $R/G/B/L$ with the individual images captured by the webcam, it is apparent that there were several instances when the webcam automatically increased the brightness in order to provide a presumably better image. However, since these automatic adjustments are not desired for consistent data acquisition, the data was manually adjusted in order to correct this behavior. Figure 5 illustrates these corrected $R/G/B/L$ values versus time.

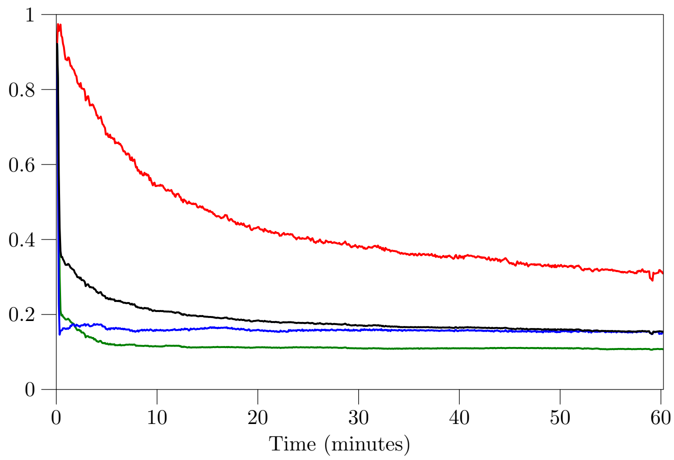

Figure 5. Repaired RGBL Values

The initial (i.e. near $t = 0$) discontinuities for $R/G/B/L$ are due to the bobbing of the tea bags which transforms the essentially clear water (i.e. $R = G = B = L = 1$) into a darker color in a small amount of time. As the bobbing ceases, the $R/G/L$ values appear to settle into their decaying patterns, whereas the B values remain essentially constant.

The data file, file `steep.d`, is available in the Support Documents of this Modeling Scenario and contains the corrected average $R/G/B/L$ values for each time t. The data file contains 5 columns, namely (t, R, G, B, L) where each t is measured in minutes and each $R/G/B/L$ is in the interval $[0, 1]$.

COMMENTS

Modeling the Data – Exponential Model

In order to determine how long the tea should steep, we first need to model L. Because L resembles a decaying exponential model, we choose to model L using the function

$$f_1(t) = A + Ce^{kt}$$

under the assumption that A is no larger than the smallest value of L. Using GNU Octave's Levenberg-Marquardt non-linear regression method `leasqr` on a subset of L (i.e. by first removing the initial apparent discontinuity formed by bobbing the tea bags), we determine that the optimal values of A, C, and k which minimize the sum of the square differences between L. Upon substituting these optimal parameter values, $f_1(t)$ then yields

$$f_1(t) = 0.153750 + 0.172515 \cdot e^{-0.095953 \cdot t}.$$

Figure 6 shows the graph of L and $f_1(t)$.

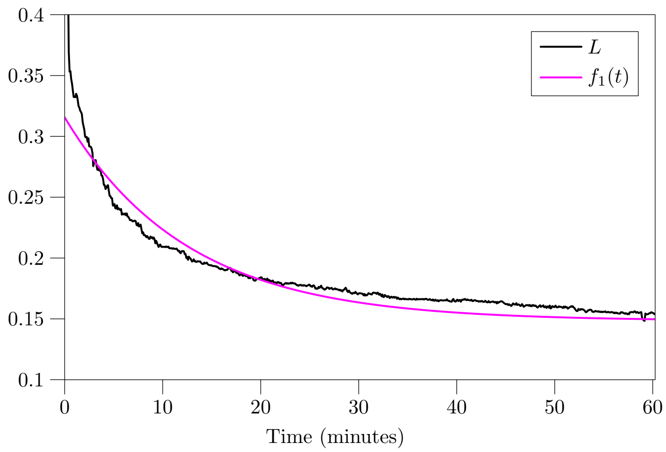

Figure 6. Modeling L with $f_1(t)$

The root mean square error between L and this "optimal" decaying exponential function is about 0.798%. Although L initially appeared to resemble a decaying exponential function, Figure 6 clearly illustrates that such a model can be improved. To understand why $f_1(t)$ fails to adequately model L, we note that $f_1(t) = A + Ce^{kt}$ is the general solution to the differential equation

$$\frac{dy}{dt} = k(y - A)$$

where k is a constant growth/decay rate. A reasonable explanation as to why this model does not adequately represent L is that k does not necessarily have to be constant. For example, if we assume that k is constant, then we are concluding that the process of heating the water (to boiling) before steeping the tea is unnecessary since the water temperature is the only component, other than the tea concentration, that is changing throughout the entire steeping process.

Model Improvement

We now consider the scenario where k depends on the temperature T of the water. Since T is changing with time t, we assume that $k = k(T) = k(T(t)) = k(t)$. Since we assume that k is not constant (with respect to T), then $k = mT + b$ is the most elementary form of a non-constant function of T. Furthermore, if we assume that T follows the Newton's Law of Cooling model (which, in our case says that $T'(t) = \lambda(T(t) - A)$ for some parameter λ and fixed ambient temperature, A),

T must be of the form $T(t) = A + Be^{\lambda t}$. Therefore, k has the form

$$\begin{aligned} k(t) &= m\left(A + Be^{\lambda t}\right) + b\,, \\ &= \alpha + \beta e^{\lambda t}\,. \end{aligned}$$

Thus, we consider (1) for modeling L

$$\frac{dy}{dt} = \left(\alpha + \beta e^{\lambda t}\right)(y - \gamma)\,, \tag{1}$$

where γ is the steady state value of L. The general solution for (1) is

$$y(t) = \gamma + Ce^{\alpha t + \frac{\beta}{\lambda} e^{\lambda t}}.$$

Using Octave's `leasqr` function to determine the parameters α, β, γ, λ, and C, minimizing the sum of square errors between observed data and model prediction gives $\alpha \approx -0.0265$, $\beta \approx -0.1886$, $\gamma \approx 0.1407$, $\lambda \approx -0.15$, and $C \approx 0.0683$. Therefore, $\frac{\beta}{\lambda} \approx 1.2577$ and so

$$f_2(t) = 0.1407 + 0.0683 \cdot e^{-0.0265 \cdot t + 1.2577 e^{-0.15 \cdot t}}$$

is the model for L approximated by GNU Octave's `leasqr` function. Since the form of $f_2(t)$ is a generalization of the form of $f_1(t)$, it is clear that $f_2(t)$ models L better than $f_1(t)$. This is demonstrated graphically in Figure 7, and the root mean square error between L and $f_2(t)$ is about 0.167% which is about $\frac{1}{5}$ of the root mean square error between L and $f_1(t)$.

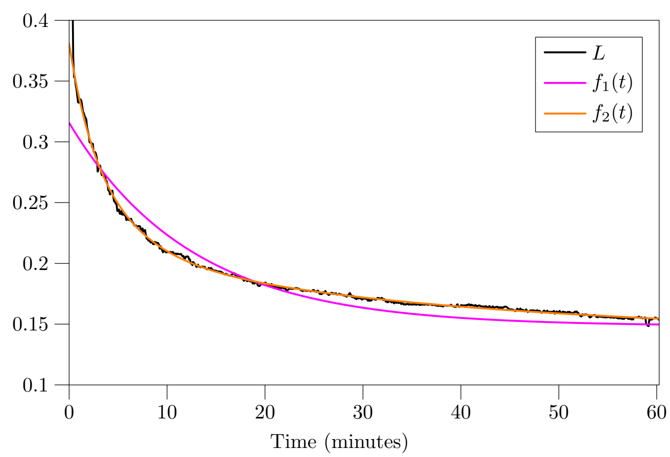

Figure 7. Modeling L with $f_2(t)$

Interpretation

We define

$$p = \frac{y(0) - y}{y(0) - \lim\limits_{\tau \to \infty} y(\tau)}$$

to be the real number in $[0, 1]$ which represents the (linear) *progress* of y toward reaching its limiting value from its initial value. For example, if $y = y(0)$, then $p = 0$. That is, initially, y has made no *progress* toward reaching its limiting value. Similarly, as y approaches its limiting value, it follows that p approaches 1. That is, in this case, y approaches complete (i.e. 100%) *progress* toward reaching its limiting value. Using this definition of p, it follows that

$$y = (1 - p) \cdot y(0) + p \cdot \lim_{\tau \to \infty} y(\tau)$$

Table 1 shows *progress* values with corresponding times t to reach each *progress*

p	t
0.5000	4.15
0.7500	12.32
0.8000	16.97
0.8500	25.18
0.9000	39.51
0.9421	60.00
0.9500	65.50
0.9800	100.03
0.9900	126.16

Table 1. Time to reach linear *progress* p

Conclusion

Question 1: *How long should the tea brew so that its concentration essentially reaches steady state?*

We conclude that the tea has reached about 94.21% saturation in an hour of steeping. If an hour is not available, then 90% saturation can be achieved in about 40 minutes.

Question 2: *How does this steady state concentration compare to the concentration attained using twice the number of tea bags brewed for the manufacturer's suggested 3-5 minutes?*

Since the tea has reached 50% saturation in about 4.15 minutes, we conclude that the manufacturer's directions of 4 tea bags steeping for 3-5 minutes yields about the same strength as allowing 2 tea bags to steep for an hour (or so). Therefore, if financial economy is more important than temporal economy, the conclusion is to use half the number of suggested tea bags and brew the tea for at least an hour.

REFERENCES

[1] Internaitonal Telecommuncation Union. 2015. Parameter values for the HDTV standards for production and international programme exchange. Recommendation ITU-R BT.709-

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